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INDECOMPOSABLE COHEN-MACAULAY MODULES AND THEIR MULTIPLICITIES

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ABSTRACT. The main aim of this paper is to find a large class of rings for which there are indecomposable maximal Cohen-Macaulay modules of arbitrary high multiplicity (or rank in the case of domains).

1. INTRODUCTION

Let $(A, \mathfrak{m})$ be a (commutative) henselian Cohen-Macaulay local ring and let $\mathrm{CM}(A)$ be the category of maximal Cohen-Macaulay A-modules (shortly MCM A-modules), i.e. of finitely generated modules M with $\operatorname{depth} M = \dim A$. For $s \in \mathbf{N}$ let $n_A(s)$ be the cardinal of isomorphism classes of indecomposable modules M from $\mathrm{CM}(A)$ whose multiplicity $e_A(M) = e(\mathfrak{m}, M) = s$. Take $n_A = \sum_{s\in\mathbf{N}} n_A(s)$.

(1.1) **First Brauer-Thrall type conjecture.** *If* $n_A = \infty$, *then* $n_A(s) \neq 0$ *for infinitely many* s.

When $\dim A = 0$ then $e_A(M) = \operatorname{length}_A(M)$ and (1.1) holds by A. V. Roiter's theorem [R, Au_1] or [P, (7.7)]. Using the Auslander-Reiten theory for MCM modules (see [Au_3, P, AR_1, Ya or Y, Appendix]) Y. Yoshino succeeded in solving positively (1.1) for reduced analytic algebras A over a perfect valued field k which are isolated singularities. Our Theorem (5.4) gives in particular the following

(1.2) **Theorem.** *Let* (A, m) *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$. *Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) A *is an isolated singularity,*
(iii) A/pA *is an isolated singularity.*

Then (1.1) *holds.*

Note that (iii) follows from (ii) when A contains a field (i.e. the equal characteristic case). When A is a domain, $e_A(M) = e(A)\cdot \operatorname{rank}(M)$ by [M_2, (14.8)]

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so in the hypothesis of Theorem (1.2) there are indecomposable MCM-modules of arbitrary high rank if $n_A = \infty$. The proof follows [Y] entirely, our contribution being mainly to extend his Lemmas (2.10) and (2.12) in the following form (see (4.8)):

(1.3) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ *and* $I_s(A)$ *the ideal defining the singular locus of* A, *i.e.* $I_s(A) = \bigcap_{q \notin \operatorname{Reg} A} q$. *Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) *if* $pA \neq 0$ *then* A_q/pA_q *is regular for every* $q \in \operatorname{Reg} A$ *containing* pA,
(iii) $I_s(A) \subseteq \mathfrak{m}$, *i.e.* A *is not regular.*

Then there exists a positive integer r *such that*

(1) *an MCM* A*-module* M *is indecomposable iff* $M/I_s(A)^r M$ *is indecomposable,*
(2) *two indecomposable MCM* A*-modules* M, N *are isomorphic iff* $M/I_s(A)^r M$ *and* $N/I_s(A)^r N$ *are isomorphic.*

In particular this theorem gives large classes of isolated singularities for which there exist Dieterich [D] reduction ideals.

In the hypothesis of (1.3) we get $n_A \leq n_{\hat{A}}$ (see (4.10)) where $\hat{A}$ is the completion of A. In particular we can improve the result from [K] and [BGS] for excellent henselian local rings (see (4.11)). Though (4.11) can also be obtained using the property of Artin approximation of excellent henselian local rings (see [Po, (1.3)] as we indicate in (5.6), we choose here an easier method (see §§3–4) which is entirely self-contained and proves to be more powerful for these questions. Our §2 contains just preliminaries arranged more or less after [Y] which we include here for completeness.

We would like to thank A. Brezuleanu and N. Radu for many helpful conversations about Theorem (4.4).

2. The singular locus of an excellent local ring

Let A be an excellent ring. Then $\operatorname{Reg} A = \{q \in \operatorname{Spec} A | A_q \text{ is regular}\}$ is an open set and $I_s(A) = \bigcap_{q \notin \operatorname{Reg} A} q$ defines the singular locus of A, i.e. $V(I_s(A)) = \operatorname{Spec} A \setminus \operatorname{Reg} A$.

(2.1) **Lemma.** *Let* $u: A \to B$ *be a flat morphism of excellent rings. Then* $I_s(B) \subset \sqrt{u(I_s(A))B}$.

Proof. If $q \in \operatorname{Reg} B$ then $q \cap A \in \operatorname{Reg} A$ by [M_1, (21.D)]. Thus a prime ideal from B containing $u(I_s(A))$ must also contain $I_s(B)$. □

(2.2) It will also be useful to express $I_s(A)$ as the radical of a certain ideal of A whose elements can be precisely described. This is already well known for rings A which are essentially of finite type over a perfect field k because in that case the Jacobian criterion for smoothness [M_1, (29.C)] applies and we have

$I_s(A) = H_{A/k}$. In general, given a finite presentation A-algebra $B = A[X]/\mathfrak{a}$, $X = (X_1, \dots, X_n)$, the nonsmooth locus of B over A is defined by the ideal

$$H_{B/A} = \sqrt{\sum_f \Delta_f((f) : \mathfrak{a})B},$$

where the sum is taken over all systems f of r-polynomials from $\mathfrak{a}$, and Δ_f is the ideal generated by all $r \times r$-minors of $\partial f/\partial Y$, $r = 1, \dots, n$, being variable (see [Po, (2.1)]). Using [Y, §2] we will present such a description of $I_s(A)$ when A is a Noetherian complete local ring having some additional properties.

(2.3) Till the end of this section $(R, \mathfrak{m})$ *is a reduced Noetherian complete local ring* with a *perfect residue field* k. Then either R contains k or R is an algebra over a Cohen ring of residue field k, i.e. a complete DVR (T, t) which is an unramified extension of $\mathbf{Z}_{(p)}$, $p := \operatorname{char} k > 0$, $t := p \cdot 1 \in T$. When R contains k we put $T := k$ and $t = 0$ in order to unify both situations.

Let $\mathscr{R}(T, R)$ be the set of all prime ideals $q \subseteq R$ for which $T \to R_q$ is a regular morphism. Clearly $\mathscr{R}(T, R) \subseteq \operatorname{Reg} R$ because T is regular and regular morphisms preserve this property [M_1, (33.B)]. When R contains k the other inclusion also holds, k being perfect. When R is in the unequal characteristic case $(pR \neq 0)$ then we suppose that

(*) $\quad R_q/pR_q$ *is regular for every* $q \in \operatorname{Reg} R$.

Thus in both situations we have $\mathscr{R}(T, R) = \operatorname{Reg} R$.

(2.4) Let $x = (x_1, \dots, x_n)$ be a system of elements from R such that (t, x) forms a system of parameters in R. Then the canonical map $T[[X]] \to R$, $X = (X_1, \dots, X_n) \to x$, is finite by Cohen Structure Theorems.

(2.5) **Lemma** (Scheja-Storch [S, (4.2)]). *There exists* x *as above such that* $\operatorname{ht}(H_{R/T[[x]]}) \geq 1$, *i.e. for every minimal prime ideal* $q \subseteq R$ *the fraction field extension* $\operatorname{Fr}(T[[x]]) \hookrightarrow R_q$ *is (finite) separable.*

Proof. When $T \neq k$ there is nothing to show because $\operatorname{char} T = 0$. Suppose $T = k$. Let $q_1, \dots, q_s$ be the minimal prime ideals of R and take an arbitrary system of parameters $y = (y_1, \dots, y_n)$ of R. If the field extensions $\alpha_i : k((y)) \to R_{q_i}$, $1 \leq i \leq s$, are all separable then $\operatorname{ht}(H_{R/k[[y]]}) \geq 1$ by the Jacobian criterion for smoothness [M_1, (29.C)]. Suppose that the $(\alpha_i)_{1 \leq i \leq e}$ are not separable for a certain e, $1 \leq e \leq s$. Then $p > 0$ and for every i, $1 \leq i \leq e$, there exists an element $z_i \in R_{q_i} \setminus k((y))$ such that $z_i^p \in k((y))$. Since α_i is finite we have

$$k((y)) \otimes_{k[[y]]} R \cong \prod_{i=1}^{s} R_{q_i}.$$

Thus we can find one $z \in R$ and $w \in k[[y]]$, $w \neq 0$, such that z/w corresponds to $(z_1, \dots, z_e, y_n, \dots, y_n)$ by the above isomorphism. Then

$h := z^p \in k[[y]]$ and $z \notin k[[y]]$. Adding a constant to z we can suppose that $h \in (y)k[[y]]$. If $h \in k[[y^p]]$ then $h \in k^p[[y^p]]$ (k is perfect) and so $z \in k[[y]]$ which is not possible.

Suppose that $h \notin k[[y_1, \dots, y_{n-1}, y_n^p]]$. After a coordinate transformation we can suppose also that h is regular in y_n. Applying the Weierstrass Preparation Theorem for $U - h$ in $k[[y, U]]$ we find a distinguished polynomial

$$(1) \qquad P = y_n^r + \sum_{i=1}^{r} a_i y_n^{r-i}, \qquad a_i \in k[[y_1, \dots, y_{n-1}, U]],\ a_i(0) = 0,$$

and an invertible formal power series $g \in k[[y, U]]$ such that

$$(2) \qquad U - h = Pg.$$

Substituting $U = h$ in P we get

$$(3) \qquad y_n^r + \sum_{i=1}^{r} a_i(y_1, \dots, y_{n-1}, h) y_n^{r-i} = 0$$

because $g(U = h) \neq 0$ since $g(0) \neq 0$ and $h(0) = 0$. Applying $\partial/\partial y_n$ in (2) we obtain

$$(\partial P/\partial y_n)g + P(\partial g/\partial y_n) = -\partial h/\partial y_n \neq 0$$

and substituting $U = h$ we get $(\partial P/\partial y_n)(U = h) \neq 0$. Thus (3) defines a separable equation for y_n over $k[[y_1, \dots, y_{n-1}, z^p]]$. In particular y_n is separable over $S := k[[y_1, \dots, y_{n-1}, z]]$. Denote $y' = (y_1, \dots, y_{n-1}, z)$. We have

$$[R_{q_i} : k((y'))]_{ins} = [R_{q_i} : k((y))]_{ins} - p$$

for every $i = 1, \dots, e$, where $[\ ]_{ins}$ denotes the inseparable degree. Repeating this procedure inductively we finally find a system of parameters x in R such that $k((x)) \hookrightarrow R_{q_i}$ is separable for every i. □

(2.6) *Remark.* If k is not perfect then the above lemma does not hold. If $a \in k\backslash k^p$ then $A = k[[X, Y]]/(X^p + aY^p)$ (after [Y, (2.7)] is a counterexample.

From now on we suppose that R is a CM-ring. Then the canonical map $T[[X]] \to R$, $X \to x$ is flat (so free) by [M_1, (36.B)].

(2.7) **Lemma.** *Let $q \in \operatorname{Reg} R$. Then there exists a system of elements x in R such that*

(i) *(t, x) is a system of parameters in R,*
(ii) *$H_{R/T[[x]]} \not\subset q$.*

Proof. If $t = 0$ then we choose a system of elements y in R which forms a regular system of parameters in R_q. If $t \in q$ then by condition (2.3)(*) we get R_q/tR_q regular. Thus there exists y such that (t, y) forms a regular system of parameters in R_q. By Lemma (2.5) there exists a system of elements z in R which forms a system of parameters $\bar{z}$ in $R/\mathfrak{a}$, $\mathfrak{a} = \sqrt{(t, y)}$ such that the map $(T/T \cap \mathfrak{a})[[\bar{z}]] \to R/\mathfrak{a}$ is generically smooth. Since q is a minimal prime ideal containing $\mathfrak{a}$ we get $(T/T\cap q)[[\bar{z}]] \to R/q$ separable and so the map $T[[y, z]] \to R_q$ is etale. Thus $x = (y, z)$ works.

Suppose now $t \notin q$. Then as above we can choose y in R which forms a regular system of parameters in R_q. Take a system of elements z in R such that (t, z) forms modulo q a system of parameters in R/q. Then (t, y, z) forms a system of parameters in R and $T[[y, z]] \to R_q$ is etale $(\operatorname{char} R/q = 0)$. Thus $x = (y, z)$ works.

(2.8) **Corollary.** $I_s(R) = \sqrt{\sum_x H_{R/T[[x]]}}$, *where the sum is taken over all systems of elements* x *such that* (t, x) *forms a system of parameters of* R.

Proof. If $q \in \operatorname{Spec} R$ does not contain $H_{R/T[[x]]}$ for a certain system x then the map $T[[x]] \to R_q$ is etale and so R_q is regular because $T[[x]]$ is so. Conversely if $q \in \operatorname{Reg} R$ then by Lemma (2.7) there exists x such that $q \not\supset H_{R/T[[x]]}$. □

(2.9) Let $S \subseteq R$ be a regular local subring such that R is a finitely generated free S-module, $R^e = R \otimes_S R$ is the *enveloping algebra* of R over S and $\mu\colon R^e \to R$ is the multiplication map. Denote $I := \operatorname{Ker}\mu$. The ideal $\mathcal{N}_S^R = \mu(\operatorname{Ann}_{R^e} I)$ is called the *Noether different* of R over S.

(2.10) **Lemma.** $\mathcal{N}_S^R \cdot \Omega_{R/S} = 0$ *and* $H_{R/S} = \sqrt{\mathcal{N}_S^R}$.

Proof. The first equality is trivial because $\Omega_{R/S} = I/I^2$. Let $q \subseteq R$ be a prime ideal. If $q \not\supset \mathcal{N}_S^R$ then $\Omega_{R_q/S} = \Omega_{R/S} \otimes_R R_q = 0$ as above. Since $S \subseteq R$ is finite free we get $S \to R_q$ etale, i.e. $q \not\supset H_{R/S}$. Conversely if $q \not\supset H_{R/S}$ then $S \to R_q$ is etale and so $\Omega_{R/S} \otimes_R R_q = 0$. Thus $I_Q = I_Q^2$ for a certain prime ideal $Q \subseteq R^e$, $Q \supseteq I$ such that $\mu(Q) = q$. By Nakayama's Lemma we get $I_Q = 0$ and so $Q \not\supset \operatorname{Ann}_{R^e} I$. Thus $\mathcal{N}_S^R \not\subset q$.

(2.11) We end this section by listing some facts from Hochschild cohomology, which can be found in [P, Chapter 11]. Let $B \subseteq A$ be an extension of rings. The nth Hochschild cohomology functors $H_B^n(A, -)$, $n \geq 0$, are defined on the category of A-bimodules with values in the category of B-modules and have the following properties:

(i) $H_B^0(A, M) = M^{(A)} := \{x \in M | ax = xa \text{ for every } a \in A\}$ for all A-bimodules M.

(ii) If M, N are two A-modules then $\operatorname{Hom}_B(M, N)$ is an A-bimodule [the left (resp. right) action of A on $\operatorname{Hom}_B(M, N)$ is given as the one induced from the action on N (resp. M)] and $H_B^0(A, \operatorname{Hom}_B(M, N)) = \operatorname{Hom}_A(M, N)$.

(iii) $H_B^1(A, M)$ is a factor A-module of $\operatorname{Der}_B(A, M) = \operatorname{Hom}_A(\Omega_{A/B}, M)$.

(iv) If A is a projective module over B and

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of A-bimodules then there exist some B-morphisms $\partial^{(n)}\colon H_B^n(A, M'') \to H_B^{n+1}(A, M')$, $n \geq 0$, such that the following sequence is exact:

$$0 \to H_B^0(A, M') \to H_B^0(A, M) \to H_B^0(A, M'') \to H_B^1(A, M')$$
$$\to \cdots \to H_B^n(A, M') \to H_B^n(A, M) \to H_B^n(A, M'') \to H_B^{n+1}(A, M') \to \cdots.$$

(2.12) **Lemma.** *Let* $S \subseteq R$ *be as in* (2.9) *and* M *an* R*-bimodule. Then* $\mathcal{N}_S^R \cdot H_S^1(R, M) = 0$.

Proof. By Lemma (2.10) we have $\mathcal{N}_S^R \Omega_{R/S} = 0$ and so $\mathcal{N}_S^R \cdot \mathrm{Hom}_R(\Omega_{R/S}, M) = 0$. Now apply (2.11)(iii). □

3. CM-APPROXIMATION

(3.1) Let R be a CM local ring and $a \subset R$ a proper ideal. The couple (A, a) is a CM-*approximation* if there exists a function $\nu: \mathbf{N} \to \mathbf{N}$ (called CM-*function*) such that for every $s \in \mathbf{N}$, every two MCM R-modules M, N and every linear R-map $\varphi: M \to N/a^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ such that $(A/a^s) \otimes_A \varphi \cong (A/a^s) \otimes_A \psi$, in other words the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/a^{\nu(s)}N \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ N & \longrightarrow & N/a^s N \end{array}$$

But given M, N then there exists ν such that for every linear R-map $\varphi: M \to N/a^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ such that $(A/a^s) \otimes_A \varphi \cong (A/a^s) \otimes_A \psi$ (this follows from a linear form of the strong approximation theorem; see e.g. [Po, (1.5)] which holds in fact in every Noetherian local ring). Thus the above definition asks for a unique function for all MCM R-modules M, N.

As we shall see in the next section the CM-approximation plays an important role in the proof of (1.3). The aim of this section is to give sufficient conditions when $(R, I_s(R))$ is a CM-approximation.

(3.2) **Lemma.** *Let* $S \subset R$ *be an extension of Noetherian rings such that* R *is a finitely generated projective module over* S, x *an element from* $\mathcal{N}_S^R$ *and* M, N *two finitely generated* R*-modules such that* M *is projective over* S. *Let* $e \in \mathbf{N}$ *be a positive integer such that* $\mathrm{Ann}_N x^e := \{z \in N | x^e z = 0\} = \mathrm{Ann}_N x^{e+1}$ *and* $s \in \mathbf{N}$. *Then for every linear* R*-map* $\varphi: M \to N/x^{e+s+1}N$ *there exists a linear* R*-map* $\psi: M \to N$ *which makes commutative the following diagram*:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/x^{e+s+1}N \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ N & \longrightarrow & N/x^{e+s} N \end{array}$$

Proof. Let $N' = \operatorname{Ann}_N x^e$. We have the following commutative diagram:

$$
(1)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & N/N' & \xrightarrow{x^{e+s+1}} & N/N' & \longrightarrow & N/N' + x^{e+s+1}N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \wr} & & \downarrow & & \\
0 & \longrightarrow & N/N' & \xrightarrow{x^{e+s}} & N/N' & \longrightarrow & N/N' + x^{e+s}N & \longrightarrow & 0
\end{array}
$$

in which the bases are exact. Indeed if $x^{e+s}z \in N'$ for a certain $z \in N$ then $x^{2e+s}z = 0$ and so $z \in \operatorname{Ann}_N x^{2e+s} = N'$. Applying the functor $\operatorname{Hom}_S(M, -)$ to (1) we get the following commutative diagram:

$$
(2)\qquad
\begin{array}{ccccccccc}
0 & \to & \operatorname{Hom}_S(M, N/N') & \to & \operatorname{Hom}_S(M, N/N') & \to & \operatorname{Hom}_S(M, N/N' + x^{e+s+1}N) & \to & 0 \\
 & & x\downarrow & & \| & & \downarrow & & \\
0 & \to & \operatorname{Hom}_S(M, N/N') & \to & \operatorname{Hom}_S(M, N/N') & \to & \operatorname{Hom}_S(M, N/N' + x^{e+s}N) & \to & 0
\end{array}
$$

where the bases are exact because M is projective over S. Clearly these bases are also exact sequences of R-bimodules and applying the Hochschild cohomology functors we get the following commutative diagram (see (2.11)(ii)):

$$
(3)\qquad
\begin{array}{ccccc}
\operatorname{Hom}_R(M, N/N') & \longrightarrow & \operatorname{Hom}_R(M, N/N' + x^{e+s+1}N) & \longrightarrow & H_S^1(R, \operatorname{Hom}_S(M, N/N')) \\
\| & & \downarrow & & \downarrow{\scriptstyle x} \\
\operatorname{Hom}_R(M, N/N') & \longrightarrow & \operatorname{Hom}_R(M, N/N' + x^{e+s}N) & \longrightarrow & H_S^1(R, \operatorname{Hom}_S(M, N/N'))
\end{array}
$$

in which the bases are exact (see (2.11)(iv)). Since the last vertical map is zero by Lemma (2.12) we get a linear R-map $\alpha: M \to N/N'$ such that the following diagram is commutative:

$$
(4)\qquad
\begin{array}{ccccc}
M & \xrightarrow{\varphi} & N/x^{e+s+1}N & \longrightarrow & N/N' + x^{e+s+1}N \\
\downarrow{\scriptstyle \alpha} & & & & \downarrow \\
N/N' & & \longrightarrow & & N/N' + x^{e+s}N
\end{array}
$$

Note that in the diagram

$$
(5)\qquad
\begin{array}{ccccc}
M & & \longrightarrow & & N/x^{e+s+1}N \\
\| & & \downarrow & & \\
M & \overset{\psi}{\dashrightarrow} & N/N' \cap x^{e+s}N & \longrightarrow & N/x^{e+s}N \\
\| & & \downarrow & & \downarrow \\
M & \longrightarrow & N/N' & \longrightarrow & N/N' + x^{e+s}N
\end{array}
$$

the small square is cartesian and so there exists ψ which makes (5) commutative.

It remains to show that $N' \cap x^{e+s}N = 0$. Indeed let $y \in N' \cap x^{e+s}N$ and $z \in N$ with $y = x^{e+s}z$. Then $0 = x^e y = x^{2e+s}z$ and so $z \in N'$, i.e. $y = x^{e+s}z = 0$. □

(3.3) **Lemma.** *Let $B \hookrightarrow A$ be a finite flat extension of Noetherian rings, $\mathfrak{a} \subset A$ an ideal and $x \in H_{A/B}$ an element. Then there exists a positive integer r such that for every finitely generated A-module N which is free over B*

$$(\mathfrak{a}N : x^r)_N = (\mathfrak{a}N : x^{r+1})_N$$

holds, where $(\mathfrak{a}N : x^r)_N = \{z \in N | x^r z \in \mathfrak{a}N\}$.

Proof. Step 1. *Reduction to the case* $(\mathfrak{a} : x) = \mathfrak{a}$. Since A is Noetherian we have $\mathfrak{a}' := (\mathfrak{a} : x^n) = (\mathfrak{a} : x^{n+1})$ for a certain positive integer n. If $xy \in \mathfrak{a}'$ for a certain $y \in A$ then $x^{n+1} y \in \mathfrak{a}$ and so $y \in \mathfrak{a}'$, i.e. $(\mathfrak{a}' : x) = \mathfrak{a}'$.

Suppose that $r' \in \mathbf{N}$ satisfies our lemma for x and $\mathfrak{a}'$. Then $r = n + r'$ works. Indeed, let N be as in our lemma. If $x^s z \in \mathfrak{a}N \subseteq \mathfrak{a}'N$ for some $s \in \mathbf{N}$ and $z \in N$ then $x^{r'} z \in \mathfrak{a}'N$ because $(\mathfrak{a}'N : x^{r'})_N = (\mathfrak{a}'N : x^{r'+1})_N$. Thus $x^r z \in x^n \mathfrak{a}'N \subseteq \mathfrak{a}N$.

Remark. $\operatorname{Ass}_A(A/\mathfrak{a}') = \{q \in \operatorname{Ass}_A(A/\mathfrak{a}) | x \notin q\}$.

Let $\mathfrak{a} = \bigcap_{i=1}^e Q_i$ be an irredundant primary decomposition of $\mathfrak{a}$, $q_i := \sqrt{Q_i}$, $q_i' := q_i \cap B$, $Q_i' := Q_i \cap B$, $\mathfrak{b} := \mathfrak{a} \cap B = \bigcap_{i=1}^e Q_i'$ and $k_i' \subseteq k_i$ the residue field extension of $B_{q_i'} \subset A_{q_i}$.

Step 2. *Case when* $k_i' = k_i$, $1 \leq i \leq e$. By Step 1 we may suppose that $(\mathfrak{a} : x) = \mathfrak{a}$. Fix an i, $1 \leq i \leq e$. Clearly $x \notin q_i$ because x is a nonzero divisor of $A/\mathfrak{a}$. Then the map $B_{q_i'} \to A_{q_i}$ is etale and so $q_i A_{q_i} = q_i' A_{q_i}$. Since $k_i' = k_i$ the extension $B_{q_i'} \subset A_{q_i}$ is dense. In particular we have

$$B_{q_i'}/Q_i' B_{q_i'} \cong A_{q_i}/Q_i' A_{q_i}$$

and it follows that $Q_i' A_{q_i} = Q_i A_{q_i}$.

We show that $r = 0$ satisfies this case. Let N be as in our lemma, and $z \in N$ such that $xz \in \mathfrak{a}N$. Then $z \in Q_i N_{q_i} = Q_i' N_{q_i}$. Thus there exists an element $y_i \in A \backslash q_i$ such that $y_i z \in Q_i' N$. Since $B/q_i' \to A/q_i$ is finite we get $(y_i A) \cap (B \backslash q_i') \neq \emptyset$. Thus changing y_i by one of its multiples we may suppose that $y_i \in B \backslash q_i'$, i.e. $z \in Q_i' N_{q_i'}$. Since N is free over B we have

$$\mathfrak{b}N = \bigcap_{j=1}^{e} Q_j' N$$

and $Q_j' N$ is exactly the q_j'-primary submodule of N associated to $\mathfrak{b}N$. Then $N \cap Q_j' N_{q_j'} = Q_j' N$ and so

$$z \in N \cap \left(\bigcap_{j=1}^{e} Q_j' N_{q_j'} \right) = \mathfrak{b}N \subseteq \mathfrak{a}N.$$

Step 3. *Case when there exists a faithfully flat B-algebra C such that for every prime ideal q associated to $C \otimes_B \mathfrak{a}$ in $D = C \otimes_B A$ the residue field extension of*

$C_{q\cap C} \hookrightarrow D_q$ *is trivial.* We apply Step 2 to the case $C \subseteq D$, $\mathfrak{a}D$, $x' = 1\otimes x \in D$. Clearly $x' \in D\otimes_A H_{A/B} \subseteq H_{D/C}$. Then there exists r such that for every finitely generated D-module N' which is free over C it follows that

$$(\mathfrak{a}N' : x'^r)_{N'} = (\mathfrak{a}N' : x'^{r+1})_{N'}.$$

Let N be a finitely generated A-module which is free over B and take $N'' = D \otimes_A N$. Then N'' is free over C and so we get in particular

$$(\mathfrak{a}N'' : x'^r)_{N''} = (\mathfrak{a}N'' : x'^{r+1})_{N''}.$$

But $(\mathfrak{a}N'' : x'^r)_{N''} = D\otimes_A(\mathfrak{a}N : x^r)_N$. Indeed, $(\mathfrak{a}N : x^r)_N$ is exactly the kernel of the composed map $f\colon N \xrightarrow{x^r} N \to N/\mathfrak{a}N$ and by flatness $\operatorname{Ker}(D\otimes_A f) \cong D\otimes_A \operatorname{Ker} f$. Thus the inclusion $u\colon (\mathfrak{a}N : x^r)_N \to (\mathfrak{a}N : x^{r+1})_N$ goes by base change in an equality. Since D is a faithfully flat A-algebra we get u surjective too.

Step 4. *General case—reduction to Step* 3. We need the following

(3.4) **Lemma.** *Let $S \subseteq R$ be a finite flat extension of Noetherian rings and denote*

$$d_{R/S} = \max_{q'\in \operatorname{Spec} S} \sum_{\substack{q\in \operatorname{Spec} R\\ q\cap S = q'}} ([k(q) : k(q')] - 1),$$

where $k(q)$ denotes the residue field of R_q. Then $d_{R/S} < \infty$ and $d_{R\otimes_S R/R} \leq d_{R/S}$ if $d_{R/S} > 0$, where the structural map $R \to R\otimes_S R$ is given by $y \to y\otimes 1$.

Applying by recurrence the above lemma we get finally a finite flat B-algebra C of the form $A\otimes_B A\otimes_B \cdots \otimes_B A$ such that $d_{C\otimes_B A/C} = 0$, i.e. $k(q\cap C) = k(q)$ for all $q \in \operatorname{Spec}(C\otimes_B A)$. Since a finite flat extension is faithfully flat we are ready. □

Proof of Lemma (3.4). Let $q' \in \operatorname{Spec} S$. Then

$$d_{R/S,q'} = \sum_{\substack{q\in \operatorname{Spec} R\\ q\cap S = q'}} ([k(q) : k(q')] - 1) < \operatorname{rank}_{k(q')} k(q')\otimes_S R,$$

the last number being bounded by the minimal number of generators of R over S. It is enough to show that

$$d_{R\otimes_S R/R,q} < d_{R/S,q'}$$

for every $q \in \operatorname{Spec} R$ lying over q' and such that $d_{R/S,q'} > 0$. So by base change we reduce the question to the case when $S = k(q') =: k$. Then R is Artinian. Let $(k_i)_{1\leq i\leq e}$ be its residue fields. It is enough to show that

$$d_{k_1\otimes_k k_i/k_1} \leq d_{k_i/k_1}, \quad 1 \leq i \leq e, \qquad d_{k_1\otimes_k k_1/k_1} < d_{k_1/k} \quad \text{if } k \neq k_1.$$

First inequality is clear because

$$1 + d_{k_1\otimes_k k_i/k_1} \leq \operatorname{rank}_{k_1} k_1\otimes_k k_i = \operatorname{rank}_k k_i = d_{k_i/k} + 1.$$

The equality holds only when $k_1 \otimes_k k_i$ is a field. But $k_1 \otimes_k k_1$ is not a field so the second inequality holds too. □

(3.5) **Lemma.** *Let $S \subseteq R$ be an extension of Noetherian rings such that R is a finitely generated projective module over S, x an element from $\mathcal{N}_S^R$ and $\mathfrak{a} \subseteq R$ an ideal. Then there exists an increasing function $\nu: \mathbf{N} \to \mathbf{N}$ such that for every $s \in \mathbf{N}$, for two finitely generated R-modules M, N which are free over S and for every linear R-map $\varphi: M \to N/(\mathfrak{a}, x^{\nu(s)})N$ there exists a linear R-map $\psi: M \to N/\mathfrak{a}N$ which makes commutative the following diagram:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/(\mathfrak{a}, x_\nu^{(s)})N \\ {\scriptstyle\psi}\vdots & & \downarrow \\ N/\mathfrak{a}N & \longrightarrow & N/(\mathfrak{a}, x^s)N \end{array}$$

Proof. Let r be the integer given by Lemma (3.3) for x and $\mathfrak{a}$. Define ν by $\nu(s) = 1 + \max\{r, s\}$. Then given M, N, s, φ as in our lemma we find the wanted ψ by applying Lemma (3.2) for x, $\overline{N} = N/\mathfrak{a}N$ and $e = r$. □

(3.6) **Lemma.** *Let $x = (x_1, \dots, x_n)$ be a system of elements from a Noetherian ring R such that for every i, $1 \le i \le n$, there exists a Noetherian subring S_i of R such that*

(i) *R is finite free over S_i,*
(ii) *$x_i \in \mathcal{N}_{S_i}^R$.*

Then there exists an increasing function $\nu: \mathbf{N} \to \mathbf{N}$ such that for every $s \in \mathbf{N}$, for two finitely generated R-modules M, N which are free over all $(S_i)_{1 \le i \le n}$ and for every linear R-map $\varphi: M \to N/x^{\nu(s)}N$ there exists a linear R-map $\psi: M \to N$ which makes commutative the following diagram:

$$(*) \qquad \begin{array}{ccc} M & \xrightarrow{\varphi} & N/x^{\nu(s)}N \\ {\scriptstyle\psi}\vdots & & \downarrow \\ N & \longrightarrow & N/x^s N \end{array}$$

Proof. Denote $\mathfrak{b}_i = (x_1, \dots, x_i)$, $i = 1, \dots, n$. Apply induction on n. If $n = 1$ then apply Lemma (3.5) for x_1 and $\mathfrak{a} = 0$. Suppose now that a function ν' is given which works for $\mathfrak{b}_{n-1}$. Let $s \in \mathbf{N}$ and ν_s'' be the function given by Lemma (3.5) for x_n and $\mathfrak{a} := \mathfrak{b}_{n-1}^{\nu'(s)}$. Define $\nu: \mathbf{N} \to \mathbf{N}$ by $\nu(s) = \nu'(s) + \nu_s''(s)$. Let M, N be two finitely generated R-modules which are free over all $(S_i)_{1 \le i \le n}$ and $\varphi: M \to N/b_n^{\nu(s)}N$ a linear R-map. Then there exists a linear R-

map $\alpha\colon M \to \overline{N} = N/\mathfrak{b}_{n-1}^{\nu'(s)}N$ which makes commutative the following diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\ \varphi\ } & N/b_n^{\nu(s)}N \\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow \\
 & & \overline{N}/x_n^{\nu_s''(s)}\overline{N} \cong N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{\nu_s''(s)})N \\
 & & \Big\downarrow \\
\overline{N} & \longrightarrow & \overline{N}/x_n^{s}\overline{N} \cong N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{s})N
\end{array}$$

Thus there exists a linear R-map $\psi\colon M \to N$ which makes commutative the following diagram:

$$\begin{array}{ccccc}
M & \xrightarrow{\ \alpha\ } & \overline{N} \cong N/b_{n-1}^{\nu'(s)}N & \longrightarrow & N/(\mathfrak{b}_{n-1}^{\nu'(s)}, x_n^{s})N \\
\Big\downarrow{\scriptstyle\psi} & & \Big\downarrow & & \Big\downarrow \\
N & \longrightarrow & N/\mathfrak{b}_{n-1}^{s}N & \longrightarrow & N/\mathfrak{b}_{n}^{s}N
\end{array}$$

Clearly ψ also makes $(*)$ commutative. □

(3.7) **Proposition.** *Let $(R, \mathfrak{m})$ be a reduced complete local CM-ring with a perfect residue field k, $p := \operatorname{char} k$ and $I_s(R)$ the ideal defining the singular locus of R. Suppose that for every $q \in \operatorname{Reg} R$ containing pR the ring R_q/pR_q is regular and $I_s(R) \subsetneq m$. Then $(R, I_s(R))$ is a CM-approximation.*

Proof. Let $T \subseteq R$ be the Cohen ring of the residue field k (see (2.3)). By Lemma (2.10) and Corollary (2.8) we have

$$I_s(R) = \sqrt{\sum_x \mathcal{N}^R_{T[[x]]}},$$

where the sum is taken over all systems of elements x such that (t, x) forms a system of parameters of R. Then we can find a system of elements $y = (y_1, \dots, y_r)$ in $I_s(R)$ such that

(1) $I_s(R) = \sqrt{yR}$,

(2) for every $i = 1, \dots, r$ there exists a system of elements $x^{(i)}$ of R such that $(t, x^{(i)}))$ forms a system of parameters of R and $y_i \in \mathcal{N}^R_{T[[x^{(i)}]]}$.

Since R is CM the inclusion $S_i := T[[x^{(i)}]] \subset R$ is finite flat (so free). Let $\nu'\colon \mathbf{N} \to \mathbf{N}$ be the function given by Lemma (3.6) for y. If M, N are two MCM R-modules then $(t, x^{(i)})$ is a regular M or N-sequence for all i. Thus M and N are finitely generated flat over S_i, $1 \le i \le r$, and so free (see [M_1, (20.C)]).

Now let u be a positive integer such that $I_s(R)^u \subset yR$ and note that ν given by $\nu(s) = u\nu'(s)$ works. □

4. CM-REDUCTION IDEALS

The aim of this section is to extend Proposition (3.7) to noncomplete rings and to apply CM-approximation in proving (1.3).

(4.1) **Lemma.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a} \subset A$ an ideal. The following statements are equivalent*:

(i) *$(A, \mathfrak{a})$ is a CM-approximation,*
(ii) *$(A, \sqrt{a})$ is a CM-approximation.*

Proof. Let u be a positive integer such that $(\sqrt{\mathfrak{a}})^u \subset \mathfrak{a}$. If (i) holds and $\nu: \mathbf{N} \to \mathbf{N}$ is the associated CM-function then as in the proof of Proposition (3.7) the function $\bar{\nu}$ given by $\bar{\nu}(s) = u\nu(s)$ works for $(A, \sqrt{\mathfrak{a}})$. If (ii) holds and $\bar{\nu}$ is the associated CM-function then the function ν given by $\nu(s) = \bar{\nu}(su)$ works. Indeed, let M, N be two MCM A-modules, $s \in \mathbf{N}$ and $\varphi: M \to N/\mathfrak{a}^{\nu(s)}N$ a linear A-map. Then there exists a linear map $\psi: M \to N$ such that the following diagram commutes:

$$\begin{array}{ccccccc} M & \xrightarrow{\varphi} & N/\mathfrak{a}^{\bar{\nu}(su)}N & \longrightarrow & N/(\sqrt{\mathfrak{a}})^{\bar{\nu}(su)}N & \longrightarrow & N/\mathfrak{a}^s N \\ {\scriptstyle\psi}\downarrow & & & & \downarrow & & \| \\ N & & \longrightarrow & & N/(\sqrt{\mathfrak{a}})^{su}N & \longrightarrow & N/\mathfrak{a}^s N \end{array}$$

(4.2) **Lemma.** *Let $A \to B$ be a flat local morphism of CM-local rings and $\mathfrak{a} \subset A$ an ideal. If $(B, \mathfrak{a}B)$ is a CM-approximation then $(A, \mathfrak{a})$ is too.*

Proof. We claim that the CM-function ν associated to $(B, \mathfrak{a}B)$ works also for $(A, \mathfrak{a})$. Indeed, let M, N be two MCM A-modules, $s \in \mathbf{N}$ and $\varphi: M \to N/\mathfrak{a}^{\nu(s)}N$ a linear A-map. Then $\overline{M} = B \otimes_A M$, $\overline{N} = B \otimes_A N$ are MCM B-modules since by flatness

$$\operatorname{depth} \overline{M} = \operatorname{depth}_A M + \operatorname{depth}(B/\mathfrak{m}B) = \operatorname{depth} A + \operatorname{depth}(B/\mathfrak{m}B) = \operatorname{depth} B$$

where $\mathfrak{m}$ denotes the maximal ideal of A (see e.g. [M_2, (23.3)]). Thus there exists a linear B-map $\bar{\psi}: \overline{M} \to \overline{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{B\otimes_A\varphi} & \overline{N}/\mathfrak{a}^{\nu(s)}\overline{N} \\ {\scriptstyle\bar{\psi}}\downarrow & & \downarrow \\ \overline{N} & \longrightarrow & \overline{N}/a^s\overline{N} \end{array}$$

Since M, N are finitely generated modules, the existence of $\bar{\psi}: \overline{M} \to \overline{N}$ such that the above diagram commutes means in other words that a certain linear system of equations L over A has a solution in B. Indeed, let $M \cong A^n/(z_1, \dots, z_e)$, $z_i = (z_{ij})_{1\le j\le n}$, $N = A^{n'}/(z'_1, \dots, z'_{e'})$, $z'_\lambda = (z'_{\lambda\mu})_{1\le\mu\le n'}$, $a = (a_1, \dots, a_v)$ a system of generators of a^s and φ is given by the matrix

$(w_{j\mu})_{1\le j\le n,\ 1\le\mu\le n'}$. Then L has the following form:

$$\sum_{j=1}^{n} z_{i\eta}X_{\eta\mu} = \sum_{\lambda=1}^{e'} Y_{i\lambda}z'_{\lambda\mu}, \qquad 1\le i\le e,\ 1\le\mu\le n',$$

$$X_{j\mu} - w_{j\mu} = \sum_{\alpha=1}^{v} a_\alpha U_{\alpha j\mu} + \sum_{\lambda=1}^{e'} Y'_{j\lambda}z'_{\lambda\mu}, \qquad 1\le j\le n.$$

Clearly $\bar{\psi}$ gives a solution of L in B. By faithfully flatness L also has a solution $(x_{j\mu}, y_{i\lambda}, y'_{j\lambda}, u_{\alpha j\mu})$ in A and the matrix $(x_{j\mu})$ defines a map $\psi: M \to N$ such that a diagram as above commutes. □

(4.3) **Proposition.** *Let $(A, \mathfrak{m})$ be an excellent local CM-ring, $p := \operatorname{char}(A/\mathfrak{m})$, and $I_s(A)$ the ideal defining the singular locus of A. Suppose that*

(i) *for every $q \in \operatorname{Reg} A$ containing pA the ring A_q/pA_q is regular,*
(ii) *there exists a flat, reduced noetherian complete local A-algebra $(B, \mathfrak{n})$ such that*
(ii_1) *$(B, \mathfrak{n})$ is CM and its residue field K is perfect,*
(ii_2) *for every $q \in \operatorname{Reg} A$ the map $A_q \to A_q \otimes_A B$ is regular,*
(iii) *$I_s(A) \subseteq \mathfrak{m}$.*

Then $(A, I_s(A))$ is a CM-approximation.

Proof. Let $q' \in \operatorname{Spec} B$ and $q := q' \cap A$. If $q \in \operatorname{Reg} A$ then $A_q \to B_{q'}$ is regular by (ii_2) and so $q' \in \operatorname{Reg} B$. Thus if $q' \not\supset I_s(A)$ then $q' \not\supset I_s(B)$, i.e. $I_s(B) \supset I_s(A)B$. Moreover $I_s(B) = \sqrt{I_s(A)B}$ by Lemma (2.1).

If q' contains pA then A_q/pA_q is regular (see (i)). Since $A_q/pA_q \to B_{q'}/pB_{q'}$ is regular by base change we get $B_{q'}/pB_{q'}$ regular too. Applying Proposition (3.7) to $(B, \mathfrak{n})$ we note that $(B, I_s(B))$ is a CM-approximation. By Lemma (4.1) $(B, I_s(A)B)$ is a CM-approximation and so $(A, I_s(A))$ is too (see Lemma (4.2)). □

(4.4) **Theorem.** *Let $(A, \mathfrak{m})$ be a reduced excellent local CM-ring, $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ and $I_s(A)$ the ideal defining the singular locus of A. Suppose that*

(i) *$[k : k^p] < \infty$ if $p > 0$,*
(ii) *for every $q \in \operatorname{Reg} A$ containing pA the ring A_q/pA_q is regular,*
(iii) *$I_s(A) \subseteq \mathfrak{m}$.*

Then $(A, I_s(A))$ is a CM-approximation.

Proof. If k is perfect then apply Proposition (4.3) for $B = \hat{A}$ the completion of $(A, \mathfrak{m})$ (the map $A \to \hat{A}$ is regular because A is excellent and $\hat{A}$ is reduced because A is so).

If k is not perfect let $K = k^{1/p^\infty}$ and let P be its prime subfield. Then from the exact sequence

$$\Gamma_{K/P} = 0 \to \Gamma_{K/k} \to \Omega_{k/P} \otimes_k K$$

we get $\operatorname{rank}_K \Gamma_{K/k} \leq \operatorname{rank}_k \Omega_{k/P} = \operatorname{rank}_k \Omega_{k/k^p} < \infty$, where $\Gamma_{K/k}$ denotes the imperfection module [M_1, (39.B)].

Using [EGA, (22.2.6)] or [NP, Corollary (3.6)] there exists a formally smooth Noetherian complete local A-algebra $(B, \mathfrak{n})$ such that

(1) $B/\mathfrak{n} \cong K$,
(2) $\dim B = \dim A + \operatorname{rank}_K \Gamma_{K/k}$.

Then the structural morphism $A \to B$ is regular by André-Radu's Theorem (see [An, BR_1, BR_2]) because A is excellent. Moreover B is a reduced CM-ring by [M_1, (33.B)]. Now apply Proposition (4.3).

(4.5) **Lemma.** *Let (A, m) be a Noetherian henselian local ring and $\mathfrak{a} \subset A$ an ideal. Suppose that $(A, \mathfrak{a})$ is a CM-approximation. Let $\nu: \mathbf{N} \to \mathbf{N}$ be its CM-function and $r = \nu(1)$. Then an MCM A-module M is indecomposable iff $M/\mathfrak{a}^r M$ is indecomposable over $A/\mathfrak{a}^r$.*

Proof (inspired by [Y, (2.10)]). Clearly $M/\mathfrak{a}^r M$ is decomposable if M is so (use Nakayama's Lemma). If M is indecomposable then $\operatorname{End}_A(M)$ is a local A-algebra, A being henselian. Let f be an idempotent from $\operatorname{End}_A(M/\mathfrak{a}^r M)$. Then there exists a linear A-map $g: M \to M$ such that $\bar{g} := (A/\mathfrak{a}) \otimes g = (A/\mathfrak{a}) \otimes f$ (ν is a CM-function). Clearly $\bar{g}$ is an idempotent. Since $\operatorname{End}_A(M)$ is local the sub-A-algebra

$$\{(A/\mathfrak{a}) \otimes h | h \in \operatorname{End}_A(M)\} \subset \operatorname{End}_A(M/\mathfrak{a}M)$$

is local too. Thus $\bar{g} = 0$ or $\bar{g} = 1$. Then $\mathfrak{a} \cdot (M/a^r M)$ contains either $\operatorname{Im} f$ or $\operatorname{Im}(1-f)$. Since f is idempotent we get either $\operatorname{Im} f = \operatorname{Im} f^r = 0$ or $\operatorname{Im}(1-f) = \operatorname{Im}(1-f)^r = 0$. Thus $f = 0$ or $f = 1$. □

(4.6) **Lemma.** *Conserving the hypothesis and the notations from* (4.5), *let M, N be two MCM A-modules such that M (resp. N) is indecomposable and $h: M \to N$ is a linear A-map. Suppose that $(A/\mathfrak{a}^r) \otimes_A h$ has a retraction* (*resp. section*). *Then h has a retraction* (*resp. section*).

Proof. Since $(A, \mathfrak{a})$ is a CM-approximation there exists a linear A-map $g: N \to M$ such that $(A/\mathfrak{a}) \otimes g$ is a retraction (resp. section) of $(A/\mathfrak{a}) \otimes h$. Then $\operatorname{Im}(1 - gh) \subseteq \mathfrak{a}M$ (resp. $\operatorname{Im}(1 - hg) \subseteq \mathfrak{a}N$). Since $\operatorname{End}_A(M)$ (resp. $\operatorname{End}_A(N)$) is a local ring we get $gh = 1 - (1 - gh)$ (resp. hg) bijective. Thus h has a retraction $(gh)^{-1} g$ (resp. a section).

(4.7) Let $\mathfrak{b}$ be an ideal in a Noetherian local ring $(B, \mathfrak{n})$. Then $\mathfrak{b}$ is a CM-*reduction ideal* if the following statements hold:

(i) An MCM B-module M is indecomposable iff $M/\mathfrak{b}M$ is indecomposable over $B/\mathfrak{b}$.

(ii) Two indecomposable MCM B-modules M, N are isomorphic iff $M/\mathfrak{b}M$ and $N/\mathfrak{b}N$ are isomorphic over $B/\mathfrak{b}$.

Note that our CM-reduction ideal is not necessarily $\mathfrak{n}$-primary as in [D]. If $\mathfrak{b}$ is a CM-reduction ideal of B then $\mathfrak{b}^s$ is also one for every $s \in \mathbf{N}$.

(4.8) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$ *and* $I_s(A)$ *the ideal defining the singular locus of* A. *Suppose that*

(i) $[k : k^p] < \infty$ *if* $p > 0$,
(ii) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular,*
(iii) $I_s(A) \subseteq \mathfrak{m}$.

Then there exists a positive integer r *such that* $I_s(A)^r$ *is a CM-reduction ideal of* A.

The proof follows from Lemmas (4.5) and (4.6).

Let n_A be the cardinal of the *isomorphism classes of indecomposable MCM A-modules.*

(4.9) **Corollary.** *Conserving the notations and hypothesis of Theorem* (4.8) *let* B *be the completion of* A *with respect to* $I_s(A)$. *Then*

(i) *An MCM A-module* M *is indecomposable iff* $B \otimes_A M$ *is an indecomposable MCM B-module*

(ii) *Two indecomposable MCM B-modules* M, N *are isomorphic iff* $B \otimes_A M$, $B \otimes_A N$ *are isomorphic over* B.

In particular $n_A \leq n_B$.

Proof. (i) By Theorem (4.8) there exists $r \in \mathbf{N}$ such that $I_s(A)^r$ is a CM-reduction ideal of A. Let M be an indecomposable MCM A-module. Then $B \otimes_A M$ is an MCM B-module by flatness and $\overline{M} := M/I_s(A)^r M$ is indecomposable over $\overline{A} := A/I_s(A)^r$. Since $\overline{A} \cong B/I_s(A)^r B$ it follows that $(B \otimes_A \overline{A}) \otimes_A M$ is indecomposable over $B \otimes_A \overline{A}$ and so $B \otimes_A M$ is indecomposable too.

Conversely if $B \otimes_A M$ is an indecomposable MCM B-module and x is a system of parameters in A then x is a $(B \otimes_A M)$-regular sequence. Since $A \to B$ is faithfully flat it follows that x is an M-regular sequence, i.e. M is an MCM A-module. Clearly M must be indecomposable because $B \otimes_A M$ is so.

(ii) If $B \otimes_A M \cong B \otimes_A N$ then $B \otimes_A \overline{M} \cong B \otimes_A \overline{N}$ and so $\overline{A} \otimes_A M \cong \overline{A} \otimes_A N$. Thus $M \cong N$ because $I_s(A)^r$ is a CM-reduction ideal of A. □

(4.10) **Corollary.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$ *and* $p := \operatorname{char} k$. *Suppose that*

(i) A *is an isolated singularity, i.e.* $\mathfrak{m} = I_s(A)$,
(ii) $[k : k^p] < \infty$ *if* $p > 0$,
(iii) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular.*

Then $n_A \leq n_{\widehat{A}}$, *where* $\widehat{A}$ *is the completion of* $(A, \mathfrak{m})$.

Remark. When k is perfect and $pA = 0$ then Corollary (4.10) is an easy consequence of [Y, (2.10), (2.12); Po, (1.3)] (see our (5.6)).

(4.11) **Corollary.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian Gorenstein local ring. Suppose that*

(i) A *is an isolated singularity of equal characteristic,*
(ii) $k := A/\mathfrak{m}$ *is algebraically closed and* $\operatorname{char} k \neq 2$,
(iii) *the completion* $\widehat{A}$ *is a simple hypersurface singularity.*

Then A *is of finite CM-type, i.e.* A *has just a finite set of isomorphic classes of indecomposable MCM* A*-modules.*

The proof follows by [GK, K, BGS] and our (4.10).

5. Bounded multiplicity CM-type

(5.1) This section is devoted to an extension of (1.2). First we list some definitions and facts from the Auslander-Reiten theory for the MCM modules (see [Au_3, AR_1, P, Ya; Y Appendix]).

Let $(A, \mathfrak{m})$ be a henselian local CM-ring and M, N two indecomposable MCM A-modules. A linear A-map $f: M \to N$ is *irreducible* if f is not an isomorphism and given any factorization $f = gh$ in the category $\mathrm{CM}(A)$, g has a section or h has a retraction. The AR-*quiver of* A is a directed graph which has as vertices the isomorphic classes of indecomposable MCM modules over A and there is an arrow from the isomorphic class of M to that of N provided there is an irreducible linear map from M to N. A *chain of irreducible maps* from M to N is a sequence of irreducible linear maps:

$$M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_n} M_n$$

with all M_i indecomposable MCM A-modules; n is called the *length of the chain.* If A is an isolated singularity then the AR-graph of A is *locally finite,* i.e. each vertex may be incident to only a finite number of other vertices (see [Au_2, AR_2, Y, (A.18)])

The following two lemmas are just variants of [Y, Lemmas (3.1), (3.2)] or [D, §1].

(5.2) **Lemma** (Harada-Sai lemma for MCM-modules). *Let* n *be a positive integer,* M_i, $0 \leq i \leq 2^n$, *some indecomposable MCM* A*-modules and* $f_i: M_{i-1} \to M_i$, $1 \leq i \leq 2^n$, *some nonisomorphic linear* A*-maps. Suppose that*

(i) $\mathfrak{m}^r$ *is a CM-reduction ideal of* A *for a certain* $r \in \mathbf{N}$,
(ii) $\operatorname{length}(M_i/\mathfrak{m}^r M_i) \leq n$, $0 \leq i \leq n$.

Then $(A/\mathfrak{m}^r) \otimes (f_{2^n} \circ \cdots \circ f_1) = 0$.

The proof follows easily from [HS, Lemma 12] and our Lemma (4.6).

(5.3) **Lemma.** *Let* n *be a positive integer,* M, N *two indecomposable MCM* A*-modules and* $\varphi: M \to N$ *a linear* A*-map. Suppose that*

(1) $\mathfrak{m}^r$ *is a CM-reduction ideal of* A *for a certain* $r \in \mathbf{N}$,
(2) $(A/\mathfrak{m}^r) \otimes \varphi \neq 0$,

(3) *there is no chain of irreducible maps from* M *to* N *of length* $< n$ *which is nontrivial modulo* $\mathfrak{m}^r$.

Then

(i) *There exist a chain of irreducible maps*

$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_n} M_n$$

and a linear A*-map* $g: M_n \to N$ *such that* $(A/\mathfrak{m}^r) \otimes (g \circ f_n \circ \cdots \circ f_1) \neq 0$.

(ii) *There exist a chain of irreducible maps*

$$N_n \xrightarrow{g_n} N_{n-1} \to \cdots \xrightarrow{g_1} N_0 = N$$

and a linear A*-map* $f: M \to N_n$ *such that* $A/\mathfrak{m}^r \otimes (g_1 \circ \cdots \circ g_n \circ f) \neq 0$.

The proof follows as in [Y, (3.2)].

(5.4) **Theorem.** *Let* $(A, \mathfrak{m})$ *be a reduced excellent henselian local CM-ring,* $k := A/\mathfrak{m}$, $p := \operatorname{char} k$, Γ *the AR-quiver of* A *and* Γ^0 *a connected component of* Γ. *Suppose that*

(i) A *is an isolated singularity,*

(ii) Γ^0 *is of bounded multiplicity type, i.e. there exists* $n \in \mathbf{N}$ *such that all indecomposable MCM modules* M *whose isomorphic classes are vertices in* Γ^0 *hold,* $e(M) \leq n$.

(iii) $[k : k^p] < \infty$ *if* $p > 0$,

(iv) *for every* $q \in \operatorname{Reg} A$ *containing* pA *the ring* A_q/pA_q *is regular.*

Then $\Gamma = \Gamma^0$ *and* Γ *is a finite graph. In particular* A *is of finite CM-type.*

Proof (inspired from [Y, (3.3)]). By (i) we have $I_s(A) = \mathfrak{m}$ and it follows that $\mathfrak{m}^r$ is a CM-reduction ideal of A for a certain $r \in \mathbf{N}$ (see Theorem (4.8)). Let x be a system of parameters of A and M an MCM A-module. By [M_2, (14.11)] we have

$$\operatorname{length}_A(M/xM) = e(xA, M)$$

because x is an M-regular sequence. Let $u \in \mathbf{N}$ be such that $\mathfrak{m}^u \subseteq xA$. By [$M_2$, (14.3), (14.4)] we get

$$e(xA, M) \leq e(\mathfrak{m}^u, M) = e(M)u^d,$$

where $d = \dim A$. Choosing x in $\mathfrak{m}^r$ it follows that

(1) $$\operatorname{length}_A(M/\mathfrak{m}^r M) \leq e(M)u^d.$$

Let $\mathscr{M}$ be the class of all MCM A-modules whose isomorphic classes are vertices in Γ^0. Using (1) we get

(2) $$\operatorname{length}_A(M/\mathfrak{m}^r M) \leq s = nu^d = \text{constant for every } M \in \mathscr{M}.$$

Let M, N be two indecomposable MCM A-modules and $f: M \to N$ a linear A-map such that $(A/\mathfrak{m}^r) \otimes f \neq 0$. If $M \in \mathscr{M}$ then there is a chain of irreducible maps from M to N of length $< t := 2^s$ which is nontrivial modulo $\mathfrak{m}^r$. Otherwise there exists a chain of irreducible maps as in Lemma (5.3)(i)

$$M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_t} M_t$$

and a linear A-map $g: M_t \to N$ such that $A/\mathfrak{m}^r \otimes (g \circ f_t \circ \cdots \circ f_1) \neq 0$. Then $A/\mathfrak{m}^r \otimes (f_t \circ \cdots \circ f_1) \neq 0$ which contradicts Lemma (5.2) (the M_i are all in Γ^0 because Γ^0 is conex and apply (2)). In particular we get $N \in \mathscr{M}$. Conversely if $N \in \mathscr{M}$ then a dual argument (using (5.3)(ii) instead of (i)) shows that $M \in \mathscr{M}$ and there exists a nontrivial chain of irreducible maps from M to N of length $< t$.

If M is a finitely generated A-module there exists a linear A-map $f: A \to M$ such that $(A/\mathfrak{m}^r) \otimes f \neq 0$ (choose $x \in M \backslash \mathfrak{m}M$ and take $f(a) = ax$). If $M \in \mathscr{M}$ then $A \in \mathscr{M}$. Moreover if M is an indecomposable A-module then $M \in \mathscr{M}$ because $A \in \mathscr{M}$. Thus $\Gamma^0 = \Gamma$. Since Γ is locally finite and every module from $\mathscr{M}$ can be connected with A by a chain of irreducible maps of length $< t$ we conclude that Γ is finite. □

(5.5) *Remark.* When A is Artinian then our theorem is a consequence of [R, Au_1]. When A is complete, $pA = 0$ and k is perfect then our theorem follows from [Y, (1.1)].

(5.6) *Remark.* Another possible approach to study the CM-type is to use Artin approximation theory (see [Ar, Po]). Let $(A, \mathfrak{m})$ be a Noetherian local ring with the property of Artin approximation (shortly A is an AP-ring), i.e. for every finite system of polynomial equations f over A, every $s \in \mathbf{N}$ and every formal solution $\hat{y}$ of f in the completion $\widehat{A}$ of A there exists a solution y of f in A such that $y \equiv \hat{y} \bmod \mathfrak{m}^s \widehat{A}$. Let M, N be two finitely generated A-modules. If A is an AP-ring then

(i) M is indecomposable iff $\widehat{A} \otimes_A M$ is so,
(ii) $M \cong N$ iff $\widehat{A} \otimes_A M \cong \widehat{A} \otimes_A N$.

For the proof note that the question can be expressed by the compatibility of some systems of polynomial equations over A (as in the proof of (4.2); but this time the equations are not linear). In particular the CM-type of A is finite if the CM-type of $\widehat{A}$ is so. Since excellent henselian local rings are AP (see [Po, Theorem (1.3)] we note that our Theorem (5.4) follows from [Y, (1.1)] when k is perfect and $pA = 0$.

Added in proof. The inequalities from (4.9) and (4.10) are in fact equalities by Elkik's algebraization theorem and (4.11) holds also when $\operatorname{char} k = 2$ (for details see, INCREST Preprint 57/1988).

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